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LETTER TO THE EDITOR

Travelling waves for a model non-linear reaction–diffusion system

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Abstract. Explicit travelling wave solutions with specified wave speeds are worked out for a model biochemical reaction proposed by Prigogine.

A biochemical trimolecular reaction model proposed by Prigogine (Nicolis and Prigogine 1977) aims to discover the types of qualitative behaviour compatible with fundamental laws such as the laws of thermodynamics and chemical kinetics. This model is described by a set of coupled non-linear partial differential equations containing reaction and diffusion terms:

$$\partial X/\partial t = A-(B+1)X+X^2Y+D_X\nabla^2 X \tag{1}$$

$$\partial Y/\partial t = BX-X^2Y+D_Y\nabla^2 Y \tag{2}$$

where ∇^2 is the Laplace operator. Here A, B, X and Y denote concentrations, D_X and D_Y are diffusion coefficients and the concentrations A and B of the reactants are variable parameters that can be controlled in the experiment.

Recently Boa (1975) studied the above system for possible steady states in a one-dimensional geometry under the simplifying assumption that the component Y diffuses very rapidly, i.e. $D_Y \to \infty, D_X < \infty$. He then obtained a non-linear second-order boundary value problem and applied phase-plane techniques to determine finite-amplitude steady-state solutions.

Our aim in this Letter is to consider this system in a one-dimensional unbounded media and to reduce the problem to one dynamical variable (X) only by assuming a large separation in the two diffusion constants involved and choosing suitable boundary conditions. On seeking a wave solution, then, we are able to reduce the system to one of Painleve type (i.e. solutions that admit only poles as movable singularities) for a special wave speed c. The general solution for this wave speed is found and from this we deduce a class of one-parameter solutions of a simple nature satisfying the boundary conditions of biological interest.

We treat the problem in a one-dimensional unbounded media. Denoting the space variable as r, the operator ∇^2 becomes d^2/dr^2. We make the assumption that $D_Y \to \infty$, $D_X < \infty$. This simplified assumption is used to reduce the problem to one dynamical

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variable only (see Boa 1975, Ibanez and Velarde 1978, Lefever *et al* 1977). The limit $D_Y \to \infty$ in equation (2) leaves the equation

$$\mathrm{d}^2 Y/\mathrm{d}r^2 = 0. \tag{3}$$

By choosing the boundary condition $Y(\infty) = Y(-\infty) = 1$, equation (3) gives the solution

$$Y = 1. \tag{4}$$

Insertion of (4) into equation (1) yields

$$D_X(\partial^2 X/\partial r^2) - (\partial X/\partial t) + X^2 - (B+1)X + A = 0. \tag{5}$$

If we let

$$\begin{aligned} X &= l\left(X' + \frac{B+1-l}{2l}\right) \\ r' &= (l/D_X)^{1/2} r \\ t' &= lt \end{aligned} \tag{6}$$

where $l = [(B+1)^2 - 4A]^{1/2}$ and drop the primes, equation (5) reduces to

$$(\partial^2 X/\partial r^2) - (\partial X/\partial t) + X^2 - X = 0. \tag{7}$$

If travelling wave solutions of equation (7) exist then X can be written in the form

$$X(r, t) = w(z) \qquad z = -ct$$

where c is the speed of propagation of the wave. Then (7) reduces to an ordinary second-order non-linear differential equation

$$\mathrm{d}^2 w/\mathrm{d}z^2 = -c(\mathrm{d}w/\mathrm{d}z) - w^2 + w. \tag{8}$$

Now we shall show that equation (8) can be reduced to one of Painleve type (Ince 1956) for a special value of c. For this value of c, we obtain explicit solutions for w satisfying the boundary conditions $w(-\infty) = 0$ and $w(\infty) = 1$ which are of biological interest.

There are 50 canonical types of ordinary differential equations whose solutions, as functions of a complex variable, have only poles as movable (i.e. dependent upon initial condition) singularities which are called Painleve type; these are enumerated by Ince (1956). It appears to be a basic property (Ablowitz and Segur 1977, Ablowitz *et al* 1978) that many of the solitons possessing non-linear partial differential equations can be reduced to ordinary differential equations of Painleve type. It therefore seems that equations with this Painleve property are somewhat simpler and are likely to be solvable explicitly. This is indeed verified for Fisher's equations by Ablowitz and Zeppetella (1979).

Make the following transformations:

$$\begin{aligned} w &= \lambda(z) W + \mu(z) \\ Z &= \phi(z) \end{aligned} \tag{9}$$

where

$$\phi = -5c^{-1} \exp(-\tfrac{1}{5}cz) \qquad \lambda = -6 \exp(-\tfrac{2}{5}cz) \qquad \mu = \tfrac{3}{25}c^2 + \tfrac{1}{2}.$$

Then, for $c = 5/\sqrt{6}$, equation (8) reduces to

$$d^2W/dZ^2 = 6W^2. \tag{10}$$

Now equation (10) is of Painleve type (Ince 1956); its solution is

$$W = \mathscr{P}(Z - k; 0, g_3) \tag{11}$$

where $\mathscr{P}(x; g_2, g_3)$ is the Weirstrauss $\mathscr{P}$ function with invariants g_2 and g_3 (Abramowitz and Stegun 1965). Here k and g_3 are arbitrary constants. Hence the solution of (8) is given by

$$W = -6 \exp[(-2/\sqrt{6})z]\mathscr{P}(-\sqrt{6} \exp(-z/\sqrt{6}) - k; 0, g_3) + 1. \tag{12}$$

In general, this solution represents a doubly periodic function with an infinite number of poles on the real axis. But by choosing $g_3 = 0$ and noting $\mathscr{P}(x; 0, 0) = x^{-2}$, we obtain the solution as

$$w = 1 - \left(1 + \frac{k}{\sqrt{6}} \exp(z/\sqrt{6})\right)^{-2}. \tag{13}$$

Thus we obtain a one-parameter family of solutions satisfying the boundary conditions $w(-\infty) = 0$ and $w(+\infty) = 1$. In order that the solutions may not blow up for any finite real z, we should have $k > 0$.

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